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TECHNICAL NOTES

A compact solution of the parallel flow three-fluid heat exchanger problem

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INTRODUCTION

Three-fluid heat exchangers are widely used in chemical processes and cryogenics. The first study of the countercurrent parallel flow three-fluid heat exchanger problem (see Fig. 1) was performed by Morley [1], though the results obtained were not given in an explicit form. Hausen [2] obtained an explicit analytical solution of temperature distributions of the same type of a heat exchanger but only for a countercurrent flow arrangement. In a number of other efforts, the same solutions for cocurrent and countercurrent flow arrangements are repeatedly reinvented [3–7]. The existing solutions for temperature distributions within the parallel flow three-fluid heat exchanger are, as a rule, of a very complex algebraic structure and frequently in a dimensional form. In addition to that, the solutions are usually not adequately tailored to be used in the same form for all possible fluid flow arrangements. Furthermore, a convenient analytical procedure for determining the so called "temperature cross" phenomenon does not exist in open literature. A temperature cross is defined to exist in an exchanger when the equalization of fluid temperatures occurs in some position(s) of the exchanger indicating reverse heat transfer has occurred, thus not fully utilizing all of the heat transfer surface.

The present note will address all these questions, and will provide explicit formulas for determining the temperature distributions of all three fluids, and in all four possible fluid flow arrangements of parallel flow three-fluid heat exchangers. In addition to that, a single analytical expression is given for determining the temperature cross for any combination of fluids involved, and all fluid flow arrangements under investigation. The results are particularly convenient for numerical computation in thermal design of a parallel flow three-fluid heat exchanger.

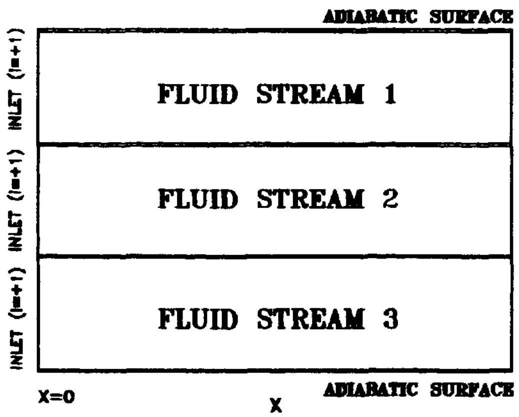

Fig. 1. Schematic of three-fluid heat exchanger.

MATHEMATICAL MODEL

In the analysis the list of idealization and approximations is as follows: (i) the three-fluid heat exchanger operates under steady-state conditions; (ii) heat exchange to the surroundings is neglected; (iii) specific heats of each fluid are constant; (iv) there are no internal thermal sources (or sinks) in the walls or fluids; (v) perfect transverse mixing occurs in each flow passage; (vi) only one of the fluids has direct thermal interactions with the other two (i.e. a three-fluid heat exchanger with two thermal communications); (vii) zero thermal conduction is assumed in fluids or in walls parallel to the fluid flow direction; (viii) heat transfer coefficients are independent of temperature, time and position; and (ix) the heat transfer area is distributed uniformly on each fluid side (the overall extended-surface temperature effectiveness is considered uniform and constant).

Four possible different parallel stream arrangements P1–P4 are identified in Table 1 by using the fluid flow indicator (see Fig. 1), for inlet side of each of the fluid streams. For example, in the cocurrent arrangement (P1), all three streams flow in the same direction. In countercurrent flow arrangements (P2–P4), one of the streams is flowing in direction opposite to other two.

The set of governing equations can be non-dimensionalized as follows:

$$i_1 \frac{d\Theta_1}{d\xi} = NTU(\Theta_2-\Theta_1), \tag{1}$$

$$i_2 \frac{d\Theta_2}{d\xi} = C^*_{1,2} NTU(\Theta_1-\Theta_2) + C^*_{1,2} R^* NTU(\Theta_3-\Theta_2), \tag{2}$$

$$i_3 \frac{d\Theta_3}{d\xi} = \frac{C^*_{1,2}}{C^*_{3,2}} R^* NTU(\Theta_2-\Theta_3), \tag{3}$$

where

$$\Theta_k = \frac{T_k - T_{1,in}}{T_{2,in}-T_{1,in}} \quad k = 1,2,3 \quad \xi = \frac{x}{L}. \tag{4}$$

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NOMENCLATURE

A	heat transfer surface area [m^2]
C^*	heat capacity rate ratio, defined by equations (5)
c_p	specific heat at constant pressure [J kg^{-1} K^{-1}]
$D_{k,j}$	coefficients in equations (8), defined in Table 4
E	functions, defined in Table 2
i	fluid flow indicator, defined in Table 1
L	flow length of the heat exchanger [m]
$L_{k,j}$	coefficients in equations (8), defined in Table 4
$\dot{m}$	mass flow rate [kg s^{-1}]
NTU	number of heat transfer units, defined by equations (5)
R^*	overall thermal resistance ratio, defined by equations (5)
s	coefficient in functions E, defined in Table 2
T	temperature [K]
$U_{k,j}$	overall heat transfer coefficient between kth and jth fluid stream based on the heat transfer area A [J s^{-1} K^{-1}]
x	axial Cartesian coordinate [m].

Greek symbols

α, β, γ	coefficients in functions Φ and Ψ, defined in Table 2
ε	heat exchanger effectiveness
ξ	non-dimensional coordinate, defined by equation (4)
Φ, Ψ	functions in the solution given by equations (6), defined in Table 2
Θ	non-dimensional temperature, defined by equation (4).

Subscripts

k, j	kth or jth fluid stream ($k, j = 1, 2, 3$)
$\xi = 0$	at $\xi = 0$
2FHE	two-fluid heat exchanger
in	at inlet

Superscripts

$*$	at temperature cross
$\rightrightarrows$	cocurrent flow arrangement
$\rightleftarrows$	countercurrent flow arrangement

$$C^*_{1,2} = \frac{(\dot{m}c_p)_1}{(\dot{m}c_p)_2} \quad C^*_{3,2} = \frac{(\dot{m}c_p)_3}{(\dot{m}c_p)_2}$$

$$NTU = \frac{(UA)_{2,1}}{(\dot{m}c_p)_1} \quad R^* = \frac{(UA)_{2,3}}{(UA)_{2,1}}. \tag{5}$$

The corresponding boundary conditions are given in Table 1.

SOLUTION

Analytical results presented in this paper were obtained by using Laplace transforms technique. Without elaborating in details the analytical procedure that can be find in any textbook of advanced engineering mathematics, only the final form of the solutions will be discussed. The set of equations (1)–(3) has been solved along with the corresponding boundary conditions (Table 1). The solutions have been obtained for any one of the four possible fluid flow arrangements with thermally unbalanced streams (see Table 1 and Fig. 1). The solutions are systematized as follows:

$$\Theta_k(\xi) = \Theta_{2,\xi=0}\Phi_k(\xi) + \Theta_{3,\xi=0}\Psi_k(\xi), \quad k = 1, 2, 3, \tag{6}$$

where the subscript k denotes the fluid flow stream. The functions $\Phi_k(\xi)$ and $\Psi_k(\xi)$ are given in Table 2. The coefficients $\Theta_{2,\xi=0}$ and $\Theta_{3,\xi=0}$ (the non-dimensionalized temperatures of fluids 2 and 3, collocated at $\xi = 0$, see Fig. 1) are defined in Table 3. The non-dimensionalized parameters $NTU, C^*_{1,2}, C^*_{3,2}, R^*$ and $\Theta_{3,in}$ are given by equations (4) and (5).

The fluid flow sign indicators i_2 and i_3 (see Table 2) should be used as given in Table 1. Note that $i_1 = +1$ always holds as adopted by the convention. Therefore, this indicator is not used explicitly in writing the solutions.

DISCUSSION

The solution obtained [equations (6)] is in a full numerical accordance with the existing solutions obtained for pure countercurrent (P2, Table 1), and cocurrent (P1, Table 1) fluid flow arrangements [5]. The comparison of this solution with the solution given in ref. [7] shows identical results for any of four arrangements. It can be proved that the explicit solution given by equations (6) can be transformed in an implicit form obtained by Baclic *et al.* [8] for any of the four analysed fluid flow arrangements. The validation of the analytical solution can also be easily verified by reducing it to those for conventional countercurrent or cocurrent two-fluid heat exchangers by setting in equation (6) $R^* = 0$ and

Table 1. The fluid flow indicator and non-dimensionalized boundary conditions for parallel stream arrangements

	Fluid flow arrangements											
	P1			P2			P3			P4		
k	i_k	ξ	$\Theta_{k,\xi}$	i_k	ξ	$\Theta_{k,\xi}$	i_k	ξ	$\Theta_{k,\xi}$	i_k	ξ	$\Theta_{k,\xi}$
1	+1	0	0	+1	0	0	+1	0	0	+1	0	0
2	+1	0	1	−1	1	1	−1	1	1	+1	0	1
3	+1	0	$\Theta_{3,in}$	+1	0	$\Theta_{3,in}$	−1	1	$\Theta_{3,in}$	−1	1	$\Theta_{3,in}$

Table 2. Functions $\Phi(\xi)$ and $\Psi(\xi)$ in equations (6)

k	$\Phi_k(\xi)$	$\Psi_k(\xi)$
1	$i_2 i_3 \frac{1}{C^*_{3,2}} \Psi_1(\xi) + \frac{1}{\gamma} E^-(\xi)$	$\frac{1}{2\alpha}\left[2 - E^+(\xi) - \frac{\beta}{\gamma} E^-(\xi)\right]$
2	$i_2 i_3 \frac{1}{C^*_{3,2}} \Psi_2(\xi) + \frac{1}{2}\left[E^+(\xi) - \frac{\beta - 2}{\gamma} E^-(\xi)\right]$	$\Psi_1(\xi) + i_2 \frac{R^* C^*_{1,2}}{\gamma} E^-(\xi)$
3	$i_3 \frac{1}{C^*_{3,2}} \{i_2[1 - \Phi_2(\xi)] - C^*_{1,2}\Phi_1(\xi)\}$	$1 - i_3 \frac{1}{C^*_{3,2}} [i_2 \Psi_2(\xi) + C^*_{1,2}\Psi_1(\xi)]$

$$E^+(\xi) = e^{s_1\xi} + e^{s_2\xi} \quad E^-(\xi) = e^{s_1\xi} - e^{s_2\xi} \quad s_1 = -(\beta - \gamma)\frac{NTU}{2} \quad s_2 = -(\beta + \gamma)\frac{NTU}{2}$$

$$\alpha = 1 + i_3 \frac{1}{C^*_{3,2}}(i_2 + C^*_{1,2}) \quad \beta = 1 + i_2 C^*_{1,2}\left[1 + R^*\left(1 + i_2 i_3 \frac{1}{C^*_{3,2}}\right)\right] \quad \gamma = \{\beta^2 - 4 i_2 R^* C^*_{1,2}\alpha\}^{1/2}$$

$C^*_{3,2} \to \infty$:

$$\lim_{\substack{R^* \to 0 \\ C^*_{3,2} \to \infty}} [\Theta_1(\xi)]_{\xi = 1} = \lim_{\substack{R^* \to 0 \\ C^*_{3,2} \to \infty}} [\Theta_{2,\xi = 0}\Phi_1(1)]$$

$$= \begin{cases} \lim\limits_{\substack{R^* \to 0 \\ C^*_{3,2} \to \infty}} [\Phi_1(1)] = \varepsilon_{\overrightarrow{2\mathrm{FHE}}}, @\, i_2 = +1, \forall\, \Theta_{3,\mathrm{in}}, i_3 \\ \lim\limits_{\substack{R^* \to 0 \\ C^*_{3,2} \to \infty}} \left[\dfrac{\Phi_1(1)}{\Phi_2(1)}\right] = \varepsilon_{\overleftrightarrow{2\mathrm{FHE}}}, @\, i_2 = -1, \forall\, \Theta_{3,\mathrm{in}}, i_3 \end{cases} \tag{7}$$

where $\varepsilon_{\overrightarrow{2\mathrm{FHE}}}$ and $\varepsilon_{\overleftrightarrow{2\mathrm{FHE}}}$ are the conventional two-fluid heat exchanger effectiveness for cocurrent and countercurrent flow arrangement, respectively [9].

In some situations, depending on the set of dimensionless parameters [see equation (5)], and boundary conditions (Table 1), the local equalization of fluid stream temperatures takes place at certain location(s) within the heat exchanger. In other words, it means that in such a situation the "temperature cross" between the temperature fields of the central and/or lateral fluids, respectively, exists. This situation can be interpreted in the sense that in a heat exchanger the inversion of the heat transfer rate directions occurs. In such a case, in a heat exchanger section where the fluid which has to be heated has larger local temperatures than the fluid which has to be cooled, the corresponding heat transfer areas are wasted. Therefore, there is an ultimate need to predict the eventual existence of the temperature cross in a heat exchanger (for design and operation parameters selected) in order to control the design.

The equalization of the corresponding local temperature values of pairs of fluid flow streams appears at the particular location in a heat exchanger ξ^* if $0 < \xi^* < 1$. Using the solution given by equations (6) one can show, after cumbersome but straightforward algebraic manipulation, that ξ^* takes values according to:

$$\xi^*_{k,j} = \frac{1}{\gamma NTU} \ln \frac{D_{k,j} + L_{k,j}}{D_{k,j} - L_{k,j}}. \tag{8}$$

Parameters $D_{k,j}$ and $L_{k,j}$ are given in Table 4. The values of $\Theta_{2,\xi=0}$ and $\Theta_{3,\xi=0}$ are defined in Table 3 (for all four possible parallel flow arrangements). Indices $\{k,j\}$ denote the corresponding temperature cross (i.e. between the fluid streams $\{1,2\}$, $\{2,3\}$ and $\{3,1\}$, respectively). It is worth noting that streams 1 and 3 are not mutually in a direct thermal contact (i.e. the fluid 1 is separated from the fluid 3 by the fluid 2). The temperature cross expression [equation (8)] is in a given form valid for all four flow arrangements, and for any of the possible three-fluid stream combinations. Therefore, this solution is more convenient than the three expressions obtained comparing subsequently the corresponding pairs of temperature distributions [8].

In Fig. 2 the nature of the temperature distributions in a three-fluid heat exchanger is demonstrated. The set of rel-

Table 3. $\Theta_{2,\xi=0}$ and $\Theta_{3,\xi=0}$, equations (6)

Arrangement	$\Theta_{2,\xi=0}$	$\Theta_{3,\xi=0}$
P1	1	$\Theta_{3,\mathrm{in}}$
P2	$\dfrac{1 - \Theta_{3,\mathrm{in}}\Psi_2(1)}{\Phi_2(1)}$	$\Theta_{3,\mathrm{in}}$
P3	$\dfrac{\Psi_3(1) - \Psi_2(1)\Theta_{3,\mathrm{in}}}{\Phi_2(1)\Psi_3(1) - \Psi_2(1)\Phi_3(1)}$	$\dfrac{\Theta_{3,\mathrm{in}} - \Theta^{P3}_{2,\xi=0}\Phi_3(1)}{\Psi_3(1)}$
P4	1	$\dfrac{\Theta_{3,\mathrm{in}} - \Phi_3(1)}{\Psi_3(1)}$

Table 4. Parameters used in equation (8)

k,j	$L_{k,j}$	$D_{k,j}$
1, 2	$\frac{1}{2}\gamma\Theta_{2,\xi=0}$	$\Theta_{2,\xi=0}\left(\frac{\beta}{2}-i_3R^*\frac{C^*_{1,2}}{C^*_{3,2}}\right)-i_2R^*C^*_{1,2}\Theta_{3,\xi=0}$
2, 3	$\frac{1}{2}\gamma(\Theta_{2,\xi=0}-\Theta_{3,\xi=0})$	$\Theta_{2,\xi=0}\left[\beta\left(\frac{1}{2}+i_2i_3\frac{1}{C^*_{3,2}}\right)-\alpha-R^*\frac{C^*_{1,2}}{C^*_{3,2}}\left(i_3+i_2\frac{1}{C^*_{3,2}}\right)\right]+\Theta_{3,\xi=0}\left[\frac{\beta}{2}-R^*C^*_{1,2}\left(i_2+i_3\frac{1}{C^*_{3,2}}\right)\right]$
3, 1	$-\frac{1}{2}\gamma\Theta_{3,\xi=0}$	$\Theta_{2,\xi=0}\left[i_2\frac{1}{C^*_{3,2}}\left(i_3\beta-R^*\frac{C^*_{1,2}}{C^*_{3,2}}\right)-\alpha\right]+\Theta_{3,\xi=0}\left(\frac{\beta}{2}-i_3R^*\frac{C^*_{1,2}}{C^*_{3,2}}\right)$

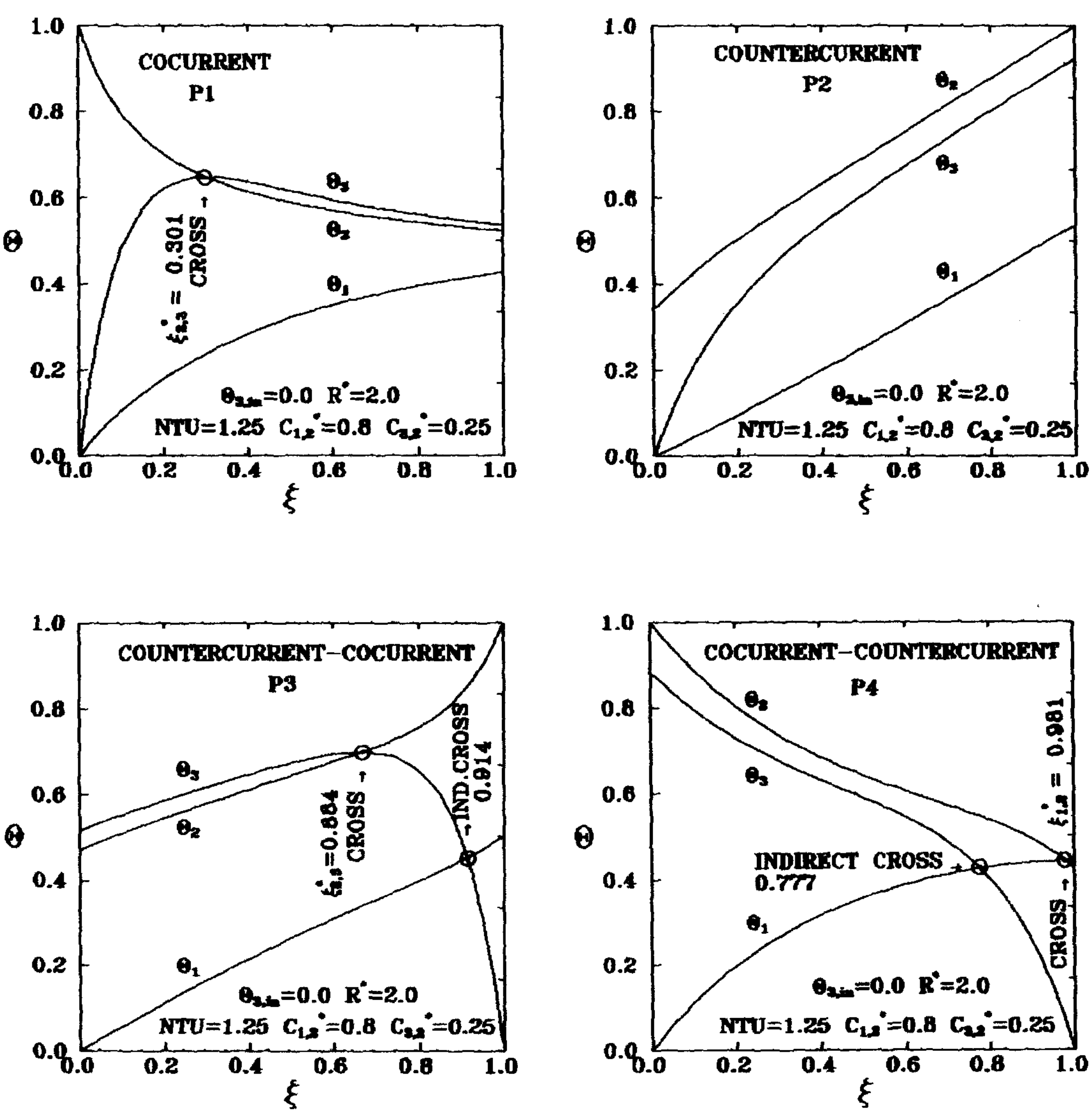

Fig. 2. Temperature distributions in a cocurrent three-fluid heat exchanger (arrangement P1), a countercurrent three-fluid heat exchanger (arrangement P2), a countercurrent–cocurrent three-fluid heat exchanger (arrangement P3), and a cocurrent–countercurrent three-fluid heat exchanger (arrangement P4).

evant parameters is the same, but the flow arrangement is different in each case. For the cocurrent flow arrangement (P1) the temperature cross is between fluids 2 and 3. In the case of the countercurrent flow arrangement the temperature cross does not exist, while for both countercurrent–cocurrent (P3) and cocurrent–countercurrent (P4) flow arrangements temperature crosses (both direct and indirect) exist. In order to determine the existence of the temperature cross without analysing the temperature distributions within a heat exchanger, one can use equation (8). It is worth noting that the calculation should include double precision.

CONCLUDING REMARK

A compact solution was obtained for the temperature distribution and temperature cross of a three-fluid heat exchanger with two thermal communications among the thermally unbalanced fluid streams. The analysis was conducted for any of four possible fluid flow arrangements.

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Conjugate convection from a sphere in a porous medium

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1. INTRODUCTION

TRANSPORT processes through porous media is a subject that has been widely studied in the scientific community during the last two decades. This interest is justified by the important role it plays in the industrial sector, particularly in the insulating systems for buildings and heat exchanger devices, energy storage systems, material processing and geothermal systems. An excellent review on this subject was recently provided by Nield and Bejan [1].

Studies of convective heat transfer from an isothermal sphere embedded in a porous medium are important in many engineering and geophysical applications such as spherical storage tanks, packed beds of spherical bodies, solidification of a magma chamber and others. However, only a little work has been devoted to this problem in the past. An early paper by Yamamoto [2] presents an analytical solution for small Rayleigh numbers. This paper has recently been extended by Sano and Okihara [3] to the case of an unsteady convective flow. But boundary-layer solution (large Rayleigh numbers) of natural convection about a general axisymmetric heated body embedded in a porous medium have been presented by several authors, notably Merkin [4], Nilson [5] and Nakayama and Koyama [6]. In particular, Cheng [7] and Chen and Chen [8] have treated the case of a sphere. It was shown in [7] that this problem admits a similarity solution. Further, a systematic analysis of the problem of natural convection from an isothermal sphere immersed in a fluid-saturated porous medium has been presented by Pop and Ingham [9]. In addition to obtaining a second-order boundary-layer solution they used a finite-difference scheme to obtain numerical results for small values of the Rayleigh numbers, as well.

However, to the authors' knowledge the conjugation features of this problem have never been analysed. It is important to mention that conjugate heat transfer problems, in which the convective heat transfer depends strongly on the thermal boundary conditions, are important in many heat transfer equipments because this dependence usually degrades the heat exchanger performance. Hence, the present problem might have some relevance to understanding of a charging or discharging process of energy in regenerative